# Study Guide and Review - Chapter 5

**Complete each identity by filling in the blank. Then name the identity.**

1. sec $\theta$ = _______

*SOLUTION:*

The reciprocal identity states sec $\theta = \frac{1}{\cos\theta}$.

2. _______ = $\frac{\sin\theta}{\cos\theta}$

*SOLUTION:*

The quotient identity states tan $\theta = \frac{\sin\theta}{\cos\theta}$.

3. _______ + 1 = $\sec^2\theta$

*SOLUTION:*

The Pythagorean identity states sec $\theta = \frac{1}{\cos\theta}$.

4. cos (90° – $\theta$) = _______

*SOLUTION:*

The cofunction identity states cos(90° − $\theta$) = sin $\theta$.

5. tan (–$\theta$) = _______

*SOLUTION:*

The odd-even identity states tan(−$\theta$) = −tan $\theta$.

6. sin ($\alpha + \beta$) = sin $\alpha$ _______ + cos $\alpha$ _______

*SOLUTION:*

The sum identity states sin ($\alpha + \beta$) = sin $\alpha$ cos $\beta$ + cos $\alpha$ sin $\beta$.

7. _______ = $\cos^2\alpha - \sin^2\alpha$

*SOLUTION:*

The double-angle identity states that cos 2$\alpha$ = $\cos^2\alpha - \sin^2\alpha$.

8. _______ = $\pm\sqrt{\frac{1+\cos\theta}{2}}$

*SOLUTION:*

The half-angle identity states cos $\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$.

9. $\frac{1-\cos 2\theta}{2}$ = _______

*SOLUTION:*

The power-reducing identity states $\frac{1-\cos 2\theta}{2} = \sin^2\theta$.

10. _______ = $\frac{1}{2}$[cos ($\alpha - \beta$) + cos ($\alpha + \beta$)]

*SOLUTION:*

The product-to-sum identity states cos $\alpha$ cos $\beta$ = $\frac{1}{2}$[cos ($\alpha - \beta$) + cos ($\alpha + \beta$)].

**Find the value of each expression using the given information.**

11. sec $\theta$ and cos $\theta$; tan $\theta$ = 3, cos $\theta$ > 0

*SOLUTION:*

Use the Pythagorean Identity that involves tan $\theta$ to find sec $\theta$.

$$\tan^2\theta + 1 = \sec^2\theta$$
$$(3)^2 + 1 = \sec^2\theta$$
$$10 = \sec^2\theta$$
$$\pm\sqrt{10} = \sec\theta$$

Since we are given that cos $\theta$ is positive, sec $\theta$ must be positive. So, sec $\theta = \sqrt{10}$. Use the reciprocal identity $\sec\theta = \frac{1}{\cos\theta}$ to find cos $\theta$.

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\sqrt{10} = \frac{1}{\cos\theta}$$
$$\cos\theta = \frac{1}{\sqrt{10}}$$
$$\cos\theta = \frac{\sqrt{10}}{10}$$

12. cot $\theta$ and sin $\theta$; cos $\theta = -\frac{1}{5}$, tan $\theta < 0$

*SOLUTION:*

Use the Pythagorean Identity that involves cos $\theta$ to find sin $\theta$.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \sin^2\theta + \left(-\frac{1}{5}\right)^2 &= 1\\ \sin^2\theta + \frac{1}{25} &= 1\\ \sin^2\theta &= \frac{24}{25}\\ \sin\theta &= \pm\frac{2\sqrt{6}}{5}\end{aligned}$$

Since tan $\theta = \frac{\sin\theta}{\cos\theta}$ is negative and cos $\theta$ is negative, sin $\theta$ must be positive. So, sin $\theta = \frac{2\sqrt{6}}{5}$.

Use the quotient identity cot $\theta = \frac{\cos\theta}{\sin\theta}$ to find cot $\theta$.

$$\begin{aligned}\cot\theta &= \frac{\cos\theta}{\sin\theta}\\ &= \frac{-\frac{1}{5}}{\frac{2\sqrt{6}}{5}}\\ &= -\frac{1}{2\sqrt{6}}\\ &= -\frac{\sqrt{6}}{12}\end{aligned}$$

13. csc $\theta$ and tan $\theta$; cos $\theta = \frac{3}{5}$, sin $\theta < 0$

*SOLUTION:*

Use the Pythagorean Identity that involves cos $\theta$ to find sin $\theta$.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1\\ \sin^2\theta + \left(\frac{3}{5}\right)^2 &= 1\\ \sin^2\theta + \frac{9}{25} &= 1\\ \sin^2\theta &= \frac{16}{25}\\ \sin\theta &= \pm\frac{4}{5}\end{aligned}$$

Since sin $\theta < 0$, $\sin\theta = -\frac{4}{5}$. Use the reciprocal identity csc $\theta = \frac{1}{\sin\theta}$ to find csc $\theta$.

$$\begin{aligned}\csc\theta &= \frac{1}{\sin\theta}\\ &= \frac{1}{-\frac{4}{5}}\\ &= -\frac{5}{4}\end{aligned}$$

Use the quotient identity tan $\theta = \frac{\sin\theta}{\cos\theta}$ to find tan $\theta$.

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{-\frac{4}{5}}{\frac{3}{5}}\\ &= -\frac{4}{3}\end{aligned}$$

14. cot $\theta$ and cos $\theta$; tan $\theta = \frac{2}{7}$, csc $\theta > 0$

*SOLUTION:*

Use the reciprocal function cot $\theta = \frac{1}{\tan\theta}$ to find cot $\theta$.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{2}{7}} = \frac{7}{2}$$

Use the Pythagorean identity that involves tan $\theta$ to find sec $\theta$.

$$\tan^2\theta + 1 = \sec^2\theta$$
$$\left(\frac{2}{7}\right)^2 + 1 = \sec^2\theta$$
$$\frac{53}{49} = \sec^2\theta$$
$$\pm\frac{\sqrt{53}}{7} = \sec\theta$$

Since csc $\theta$ is positive, sin $\theta$ is positive. Since tan $\theta = \frac{\sin\theta}{\cos\theta}$ is positive and sin $\theta$ is positive, cos $\theta$ has to be positive. Since cos $\theta$ is positive, sec $\theta$ is positive. So, sec $\theta = \frac{\sqrt{53}}{7}$. Use the reciprocal identity cos $\theta = \frac{1}{\sec\theta}$ to find cos $\theta$.

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{\sqrt{53}}{7}} = \frac{7}{\sqrt{53}} = \frac{7\sqrt{53}}{53}$$

15. sec $\theta$ and sin $\theta$; cot $\theta = -2$, csc $\theta < 0$

*SOLUTION:*

Use the Pythagorean identity that involves cot $\theta$ to find csc $\theta$.

$$\cot^2\theta + 1 = \csc^2\theta$$
$$(-2)^2 + 1 = \csc^2\theta$$
$$5 = \csc^2\theta$$
$$\pm\sqrt{5} = \csc\theta$$

Since csc $\theta < 0$, csc $= -\sqrt{5}$. Use the reciprocal identity sin $\theta = \frac{1}{\csc\theta}$ to find sin $\theta$.

$$\sin\theta = \frac{1}{\csc\theta} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

Use the Pythagorean identity that involves sin $\theta$ to find cos $\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(-\frac{\sqrt{5}}{5}\right)^2 + \cos^2\theta = 1$$
$$\frac{1}{5} + \cos^2\theta = 1$$
$$\cos^2\theta = \frac{4}{5}$$
$$\cos\theta = \pm\frac{2}{\sqrt{5}}$$
$$\cos\theta = \pm\frac{2\sqrt{5}}{5}$$

Since cot $\theta = \frac{\cos\theta}{\sin\theta}$ is negative and sin $\theta$ is negative, cos $\theta$ must be positive. So, cos $\theta = \frac{2\sqrt{5}}{5}$. Use the reciprocal identity sec $\theta = \frac{1}{\cos\theta}$ to find sec $\theta$.

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{2\sqrt{5}}{5}} = \frac{5}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$$

16. cos $\theta$ and sin $\theta$; cot $\theta$ = $\frac{3}{8}$, sec $\theta$ < 0

*SOLUTION:*

Use the Pythagorean identity that involves cot to find csc $\theta$.

$$\cot^2\theta + 1 = \csc^2\theta$$
$$\left(\frac{3}{8}\right)^2 + 1 = \csc^2\theta$$
$$\frac{73}{64} = \csc^2\theta$$
$$\pm\frac{\sqrt{73}}{8} = \csc\theta$$

Since sec $\theta$ < 0, cos $\theta$ is negative. Since cot $\theta$ = $\frac{\cos\theta}{\sin\theta}$ is positive and cos $\theta$ is negative, sin $\theta$ is negative. Since sin $\theta$ is negative, csc $\theta$ must be negative. So, $\csc\theta = -\frac{\sqrt{73}}{8}$. Use the reciprocal identity sin $\theta$ = $\frac{1}{\csc\theta}$ to find sin $\theta$.

$$\begin{aligned}\sin\theta &= \frac{1}{\csc\theta}\\ &= \frac{1}{-\frac{\sqrt{73}}{8}}\\ &= -\frac{8}{\sqrt{73}}\\ &= -\frac{8\sqrt{73}}{73}\end{aligned}$$

Use the Pythagorean identity that involves sin $\theta$ to find cos $\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(-\frac{8\sqrt{73}}{73}\right)^2 + \cos^2\theta = 1$$
$$\frac{64}{73} + \cos^2\theta = 1$$
$$\cos^2\theta = \frac{9}{73}$$
$$\cos\theta = \pm\frac{3}{\sqrt{73}}$$
$$\cos\theta = \pm\frac{3\sqrt{73}}{73}$$

It was determined that cos $\theta$ is negative. So,

$$\cos\theta = -\frac{3}{\sqrt{73}} \text{ or } -\frac{3\sqrt{73}}{73}$$

**Simplify each expression.**

17. $\sin^2(-x) + \cos^2(-x)$

*SOLUTION:*

$$\begin{aligned}\sin^2(-x)+\cos^2(-x) &= [\sin(-x)]^2 + [\cos(-x)]^2\\ &= (-\sin x)^2 + (\cos x)^2\\ &= (-1\cdot\sin x)^2 + \cos^2 x\\ &= [(-1)^2(\sin x)^2] + \cos^2 x\\ &= (\sin x)^2 + \cos^2 x\\ &= \sin^2 x + \cos^2 x\\ &= 1\end{aligned}$$

18. $\sin^2 x + \cos^2 x + \cot^2 x$

*SOLUTION:*

$$\begin{aligned}\sin^2 x + \cos^2 x + \cot^2 x &= \sin^2 x + \cos^2 x + \cot^2 x\\ &= 1 + \cot^2 x\\ &= \csc^2 x\end{aligned}$$

19. $\frac{\sec^2 x - \tan^2 x}{\cos(-x)}$

*SOLUTION:*

$$\begin{aligned}\frac{\sec^2 x - \tan^2 x}{\cos(-x)} &= \frac{\sec^2 x - \tan^2 x}{\cos x}\\ &= \frac{1}{\cos x}\\ &= \sec x\end{aligned}$$

20. $\frac{\sec^2 x}{\tan^2 x + 1}$

*SOLUTION:*

$$\begin{aligned}\frac{\sec^2 x}{\tan^2 x + 1} &= \frac{\sec^2 x}{\sec^2 x}\\ &= 1\end{aligned}$$

21. $\frac{1}{1-\sin x}$

*SOLUTION:*

$$\begin{aligned}\frac{1}{1-\sin x}&=\frac{1+\sin x}{1-\sin^2 x}\\&=\frac{1+\sin x}{\cos^2 x}\\&=\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\\&=\frac{1}{\cos^2 x}+\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\\&=\sec^2 x+\tan x\sec x\end{aligned}$$

22. $\frac{\cos x}{1+\sec x}$

*SOLUTION:*

$$\begin{aligned}\frac{\cos x}{1+\sec x}&=\frac{\cos x(1-\sec x)}{1-\sec^2 x}\\&=\frac{\cos x(1-\sec x)}{-\tan^2 x}\\&=\frac{\cos x-\cos x\sec x}{-\tan^2 x}\\&=\frac{\cos x}{-\tan^2 x}-\frac{\cos x\sec x}{-\tan^2 x}\\&=\frac{\cos x}{-\frac{\sin^2 x}{\cos^2 x}}-\frac{\cos x\sec x}{-\frac{\sin^2 x}{\cos^2 x}}\\&=-\frac{\cos^3 x}{\sin^2 x}+\frac{\cos^3 x\sec x}{\sin^2 x}\\&=-\cos x\frac{\cos^2 x}{\sin^2 x}+\frac{\cos^3 x\frac{1}{\cos x}}{\sin^2 x}\\&=-\cos x\cot^2 x+\frac{\cos^2 x}{\sin^2 x}\\&=-\cos x\cot^2 x+\cot^2 x\\&=\cot^2 x-\cos x\cot^2 x\end{aligned}$$

**Verify each identity.**

23. $\frac{\sin\theta}{1-\cos\theta}+\frac{\sin\theta}{1+\cos\theta}=2\csc\theta$

*SOLUTION:*

$\frac{\sin\theta}{1-\cos\theta}+\frac{\sin\theta}{1+\cos\theta}$

$=\frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta}+\frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$ Common denominator

$=\frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}+\frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$ Pythagorean Identity

$=\frac{1+\cos\theta}{\sin\theta}+\frac{1-\cos\theta}{\sin\theta}$ Divide out common factor of $\sin\theta$

$=\frac{2}{\sin\theta}$ Add.

$=2\csc\theta$ Reciprocal Identity

24. $\frac{\cos\theta}{\sec\theta}+\frac{\sin\theta}{\csc\theta}=1$

*SOLUTION:*

$\frac{\cos\theta}{\sec\theta}+\frac{\sin\theta}{\csc\theta}$

$=\cos\theta\cos\theta+\sin\theta\sin\theta$ Reciprocal Identities

$=\cos^2\theta+\sin^2\theta$ Simplify.

$=1$ Pythagorean Identity

25. $\frac{\cot\theta}{1+\csc\theta}+\frac{1+\csc\theta}{\cot\theta}=2\sec\theta$

*SOLUTION:*

$\frac{\cot\theta}{1+\csc\theta}+\frac{1+\csc\theta}{\cot\theta}$

$=\frac{\cot^2\theta}{\cot\theta(1+\csc\theta)}+\frac{(1+\csc\theta)^2}{\cot\theta(1+\csc\theta)}$ Common denominator

$=\frac{\cot^2\theta+(1+\csc\theta)^2}{\cot\theta(1+\csc\theta)}$ Add.

$=\frac{\csc^2\theta-1+1+2\csc\theta+\csc^2\theta}{\cot\theta(1+\csc\theta)}$ Expand and use Pythagorean Identity.

$=\frac{2\csc^2\theta+2\csc\theta}{\cot\theta(1+\csc\theta)}$ Simplify.

$=\frac{2\csc\theta(\csc\theta+1)}{\cot\theta(1+\csc\theta)}$ Factor.

$=\frac{2\csc\theta}{\cot\theta}$ Divide out common factor.

$=\frac{2}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}$ Reciprocal/Quotient Identities

$=\frac{2}{\cos\theta}$ Multiply.

$=2\sec\theta$ Reciprocal Identity

26. $\frac{\cos\theta}{1-\sin\theta}=\frac{1+\sin\theta}{\cos\theta}$

*SOLUTION:*

$\frac{\cos\theta}{1-\sin\theta}$

$=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$ Multiply by conjugate, $1+\sin\theta$.

$=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$ Pythagorean Identity.

$=\frac{1+\sin\theta}{\cos\theta}$ Divide out the common factor $\cos\theta$.

27. $\dfrac{\cot^2\theta}{1+\csc\theta} = \csc\theta - 1$

*SOLUTION:*

| | |
|---|---|
| $\dfrac{\cot^2\theta}{1+\csc\theta}$ | |
| $=\dfrac{\cot^2\theta(1-\csc\theta)}{(1+\csc\theta)(1-\csc\theta)}$ | Multiply by conjugate, $1-\csc\theta$. |
| $=\dfrac{\cot^2\theta(1-\csc\theta)}{1-\csc^2\theta}$ | Simplify. |
| $=\dfrac{\cot^2\theta(1-\csc\theta)}{-\cot^2\theta}$ | Pythagorean Identity |
| $=-(1-\csc\theta)$ | Divide out the common factor of $\cot^2\theta$. |
| $=\csc\theta-1$ | Simplify. |

28. $\dfrac{\sec\theta}{\tan\theta}+\dfrac{\csc\theta}{\cot\theta} = \sec\theta + \csc\theta$

*SOLUTION:*

| | |
|---|---|
| $\dfrac{\sec\theta}{\tan\theta}+\dfrac{\csc\theta}{\cot\theta}$ | |
| $=\dfrac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}+\dfrac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$ | Reciprocal/Quotient Identities |
| $=\dfrac{1}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta}+\dfrac{1}{\sin\theta}\cdot\dfrac{\sin\theta}{\cos\theta}$ | Multiply by reciprocals. |
| $=\dfrac{1}{\sin\theta}+\dfrac{1}{\cos\theta}$ | Multiply. |
| $=\csc\theta+\sec\theta$ | Reciprocal Identities |

29. $\dfrac{\sec\theta+\csc\theta}{1+\tan\theta} = \csc\theta$

*SOLUTION:*

| | |
|---|---|
| $\dfrac{\sec\theta+\csc\theta}{1+\tan\theta}$ | |
| $=\dfrac{\frac{1}{\cos\theta}+\frac{1}{\sin\theta}}{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}$ | Use Reciprocal and Quotient Identities and write 1 as $\frac{\cos\theta}{\cos\theta}$. |
| $=\dfrac{\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}}{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}$ | Add fractions in numerator. |
| $=\dfrac{\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}}{\frac{\cos\theta+\sin\theta}{\cos\theta}}$ | Add fractions in denominator. |
| $=\dfrac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta+\cos\theta}$ | Multiply by the reciprocal. |
| $=\dfrac{1}{\sin\theta}$ | Multiply. |
| $=\csc\theta$ | Reciprocal Identity |

30. $\cot\theta\csc\theta + \sec\theta = \csc^2\theta\sec\theta$

*SOLUTION:*

| | |
|---|---|
| $\cot\theta\csc\theta+\sec\theta$ | |
| $=\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{1}{\sin\theta}+\dfrac{1}{\cos\theta}$ | Quotient and Reciprocal Identities |
| $=\dfrac{\cos\theta}{\sin^2\theta}+\dfrac{1}{\cos\theta}$ | Multiply. |
| $=\dfrac{\cos^2\theta}{\sin^2\theta\cos\theta}+\dfrac{\sin^2\theta}{\sin^2\theta\cos\theta}$ | Common denominator. |
| $=\dfrac{\cos^2\theta+\sin^2\theta}{\sin^2\theta\cos\theta}$ | Add. |
| $=\dfrac{1}{\sin^2\theta\cos\theta}$ | Pythagorean Identity |
| $=\csc^2\theta\sec\theta$ | Reciprocal Identities |

31. $\dfrac{\sin\theta}{\sin\theta+\cos\theta} = \dfrac{\tan\theta}{1+\tan\theta}$

*SOLUTION:*

| | |
|---|---|
| $\dfrac{\sin\theta}{\sin\theta+\cos\theta}$ | |
| $=\dfrac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\cos\theta}}$ | Multiply numerator and denominator by $\cos\theta$. |
| $=\dfrac{\tan\theta}{1+\tan\theta}$ | Use the Quotient Identity and simplify. |

32. $\cos^4\theta - \sin^4\theta = \dfrac{1-\tan^2\theta}{\sec^2\theta}$

*SOLUTION:*

| | |
|---|---|
| $\cos^4\theta-\sin^4\theta$ | |
| $=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)$ | Factor. |
| $=(1)\left(\dfrac{1}{\sec^2\theta}-\dfrac{1}{\csc^2\theta}\right)$ | Pythagorean and Reciprocal Identities |
| $=\dfrac{1}{\sec^2\theta}\left(1-\dfrac{\sec^2\theta}{\csc^2\theta}\right)$ | Factor $\frac{1}{\sec^2\theta}$ from each term. |
| $=\dfrac{1}{\sec^2\theta}\left(1-\dfrac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}\right)$ | Reciprocal Identities |
| $=\dfrac{1}{\sec^2\theta}(1-\tan^2\theta)$ | Divide and use the Quotient Identity. |
| $=\dfrac{1-\tan^2\theta}{\sec^2\theta}$ | Multiply. |

**Find all solutions of each equation on the interval [0, 2π].**

33. $2\sin x = \sqrt{2}$

*SOLUTION:*

$$2\sin x=\sqrt{2}$$
$$\sin x=\frac{\sqrt{2}}{2}$$

On the interval [0, 2π), $\sin x=\frac{\sqrt{2}}{2}$ when $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

34. $4\cos^2 x = 3$

*SOLUTION:*

$$\begin{aligned} 4\cos^2 x &= 3 \\ \cos^2 x &= \frac{3}{4} \\ \cos x &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

On the interval [0, 2π), $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{11\pi}{6}$ and $\cos x = -\frac{\sqrt{3}}{2}$ when $x = \frac{5\pi}{6}$ and $x = \frac{7\pi}{6}$.

35. $\tan^2 x - 3 = 0$

*SOLUTION:*

$\tan^2 x - 3 = 0$

$$\begin{aligned} \tan^2 x - 3 &= 0 \\ \tan^2 x &= 3 \\ \tan x &= \pm\sqrt{3} \end{aligned}$$

On the interval [0, 2π), $\tan x = \sqrt{3}$ when $x = \frac{\pi}{3}$ and $x = \frac{4\pi}{3}$ and $\tan x = -\sqrt{3}$ when $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

36. $9 + \cot^2 x = 12$

*SOLUTION:*

$$\begin{aligned} 9 + \cot^2 x &= 12 \\ \cot^2 x &= 3 \\ \cot x &= \pm\sqrt{3} \end{aligned}$$

On the interval [0, 2π), $\cot x = \sqrt{3}$ when $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$ and $\cot x = -\sqrt{3}$ when $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$.

37. $2\sin^2 x = \sin x$

*SOLUTION:*

$$\begin{aligned} 2\sin^2 x &= \sin x \\ 2\sin^2 x - \sin x &= 0 \\ \sin x(2\sin x - 1) &= 0 \end{aligned}$$

$$\sin x = 0$$

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \end{aligned}$$

On the interval [0, 2π), $\sin x = 0$ when $x = 0$ and $\pi$ and $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

38. $3\cos x + 3 = \sin^2 x$

*SOLUTION:*

$$\begin{aligned} 3\cos x + 3 &= \sin^2 x \\ 3\cos x + 3 - \sin^2 x &= 0 \\ 3\cos x + 3 - (1 - \cos^2 x) &= 0 \\ \cos^2 x + 3\cos x + 2 &= 0 \\ (\cos x + 1)(\cos x + 2) &= 0 \end{aligned}$$

$$\begin{aligned} \cos x + 1 &= 0 \\ \cos x &= -1 \end{aligned}$$

$$\begin{aligned} \cos x + 2 &= 0 \\ \cos x &= -2 \end{aligned}$$

On the interval [0, 2π), $\cos x = -1$ when $x = \pi$. The equation $\cos x = -2$ has no solution since the minimum value the cosine function can attain is −1.

**Solve each equation for all values of *x*.**

39. $\sin^2 x - \sin x = 0$

*SOLUTION:*

$$\sin^2 x - \sin x = 0$$
$$\sin x(\sin x - 1) = 0$$

$$\sin x = 0$$

$$\sin x - 1 = 0$$
$$\sin x = 1$$

The period of sine is $2\pi$, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to $\sin x = 0$ on this interval are 0 and $\pi$ and the solution to $\sin x = 1$ on this interval is $\frac{\pi}{2}$.

Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of $2\pi$. The solutions $x = 0 + 2n\pi$ and $x = \pi + 2n\pi$ can be combined to $x = n\pi$. Therefore, the general form of the solutions is $n\pi$, $\frac{\pi}{2} + 2n\pi$, where $n$ is an integer.

40. $\tan^2 x = \tan x$

*SOLUTION:*

$$\tan^2 x = \tan x$$
$$\tan^2 x - \tan x = 0$$
$$\tan x(\tan x - 1) = 0$$

$$\tan x = 0$$

$$\tan x - 1 = 0$$
$$\tan x = 1$$

The period of tangent is $\pi$, so you only need to find solutions on the interval $[0, \pi)$. The solution to $\tan x = 0$ on this interval is 0 and the solution to $\tan x = 1$ on this interval is $\frac{\pi}{4}$.

Solutions on the interval $(-\infty, \infty)$ are found by adding integer multiples of $\pi$. Therefore, the general form of the solutions is $n\pi$, $\frac{\pi}{4} + n\pi$, where $n$ is an integer.

41. $3 \cos x = \cos x - 1$

*SOLUTION:*

$$3\cos x = \cos x - 1$$
$$2\cos x = -1$$
$$\cos x = -\frac{1}{2}$$

The period of cosine is $2\pi$, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to $\cos x = -\frac{1}{2}$ on this interval are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Solutions on the interval $(-\infty, \infty)$ are found by adding integer multiples of $2\pi$. Therefore, the general form of the solutions is $\frac{2\pi}{3} + 2n\pi$, $\frac{4\pi}{3} + 2n\pi$, where $n$ is an integer.

42. $\sin^2 x = \sin x + 2$

*SOLUTION:*

$$\sin^2 x = \sin x + 2$$
$$\sin^2 x - \sin x - 2 = 0$$
$$(\sin x - 2)(\sin x + 1) = 0$$

$$\sin x - 2 = 0$$
$$\sin x = 2$$

$$\sin x + 1 = 0$$
$$\sin x = -1$$

The period of sine is $2\pi$, so you only need to find solutions on the interval $[0, 2\pi)$. The equation $\sin x = 2$ has no solution since the maximum value the sine function can attain is 1. The solution to $\sin x = -1$ on this interval is $\frac{3\pi}{2}$.

Solutions on the interval $(-\infty, \infty)$ are found by adding integer multiples of $2\pi$. Therefore, the general form of the solutions is $\frac{3\pi}{2} + 2n\pi$, where $n$ is an integer.

43. $\sin^2 x = 1 - \cos x$

*SOLUTION:*

$$\sin^2 x = 1 - \cos x$$
$$1 - \cos^2 x = 1 - \cos x$$
$$\cos^2 x - \cos x = 0$$
$$\cos x(\cos x - 1) = 0$$

$$\cos x = 0$$

$$\cos x - 1 = 0$$
$$\cos x = 1$$

The period of cosine is $2\pi$, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to cos $x = 0$ on this interval are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and the solution to cos $x = 1$ on this interval is 0.

Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of $2\pi$.

The solutions $x = \frac{\pi}{2} + 2n\pi$ and $x = \frac{3\pi}{2} + 2n\pi$ can be combined to $x = \frac{\pi}{2} + n\pi$.

Therefore, the general form of the solutions is $2n\pi$, $\frac{\pi}{2} + n\pi$, where $n$ is an integer.

44. $\sin x = \cos x + 1$

*SOLUTION:*

$$\sin x = \cos x + 1$$
$$(\sin x)^2 = (\cos x + 1)^2$$
$$\sin^2 x = \cos^2 x + 2\cos x + 1$$
$$1 - \cos^2 x = \cos^2 x + 2\cos x + 1$$
$$2\cos^2 x + 2\cos x = 0$$
$$2\cos x(\cos x + 1) = 0$$

$$2\cos x = 0$$
$$\cos x = 0$$

$$\cos x + 1 = 0$$
$$\cos x = -1$$

The period of cosine is $2\pi$, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to cos $x = 0$ on this interval are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and the solution to cos $x = -1$ on this interval is $\pi$. Since each side of the equation was squared, check for extraneous solutions.

$$\sin x = \cos x + 1$$
$$\sin\left(\frac{\pi}{2}\right) \stackrel{?}{=} \cos\left(\frac{\pi}{2}\right) + 1$$
$$1 = 1$$
$$\sin x = \cos x + 1$$
$$\sin\left(\frac{3\pi}{2}\right) \stackrel{?}{=} \cos\left(\frac{3\pi}{2}\right) + 1$$
$$-1 \neq 1$$
$$\sin x = \cos x + 1$$
$$\sin \pi \stackrel{?}{=} \cos \pi + 1$$
$$0 = 0$$

Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of $2\pi$. Therefore, the general form of the solutions is $\frac{\pi}{2} + 2n\pi$, $\pi + 2n\pi$, where $n$ is an integer.

**Find the exact value of each trigonometric expression.**

45. $\cos 15°$

*SOLUTION:*

Write 15° as the sum or difference of angle measures with cosines that you know.

$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

46. $\sin 345°$

*SOLUTION:*

Write 345° as the sum or difference of angle measures with sines that you know.

$$\begin{aligned}\sin 345° &= \sin(300° + 45°)\\ &= \sin 300° \cos 45° + \cos 300° \sin 45°\\ &= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= -\frac{\sqrt{6}-\sqrt{2}}{4}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

47. $\tan \frac{13\pi}{12}$

*SOLUTION:*

$$\begin{aligned}\tan\frac{13\pi}{12} &= \tan\left(\frac{5\pi}{6} + \frac{\pi}{4}\right)\\ &= \frac{\tan\frac{5\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{5\pi}{6}\tan\frac{\pi}{4}}\\ &= \frac{-\frac{\sqrt{3}}{3} + 1}{1 - \left(-\frac{\sqrt{3}}{3}\right)(1)}\\ &= \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}\\ &= \frac{3-\sqrt{3}}{3+\sqrt{3}}\\ &= \frac{3-\sqrt{3}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}}\\ &= \frac{9-6\sqrt{3}+3}{9-3}\\ &= \frac{12-6\sqrt{3}}{6}\\ &= 2-\sqrt{3}\end{aligned}$$

48. $\sin \frac{7\pi}{12}$

*SOLUTION:*

Write $\frac{7\pi}{12}$ as the sum or difference of angle measures with sines that you know.

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$
$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

49. $\cos -\frac{11\pi}{12}$

*SOLUTION:*

Write $-\frac{11\pi}{12}$ as the sum or difference of angle measures with cosines that you know.

$$\cos -\frac{11\pi}{12} = \cos\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right)$$
$$= \cos -\frac{\pi}{6}\cos\frac{3\pi}{4} + \sin -\frac{\pi}{6}\sin\frac{3\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\cdot -\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2}$$
$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{-\sqrt{6}-\sqrt{2}}{4}$$

50. $\tan \frac{5\pi}{12}$

*SOLUTION:*

Write $\frac{5\pi}{12}$ as the sum or difference of angle measures with tangents that you know.

$$\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$
$$= \frac{\tan\frac{\pi}{6} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{6}\tan\frac{\pi}{4}}$$
$$= \frac{\frac{\sqrt{3}}{3} + 1}{1 - \frac{\sqrt{3}}{3}(1)}$$
$$= \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}}$$
$$= \frac{3+\sqrt{3}}{3-\sqrt{3}}$$
$$= \frac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}}$$
$$= \frac{12+6\sqrt{3}}{6}$$
$$= 2+\sqrt{3}$$

**Simplify each expression.**

51. $\dfrac{\tan\frac{\pi}{9} + \tan\frac{8\pi}{9}}{1 - \tan\frac{\pi}{9}\tan\frac{8\pi}{9}}$

*SOLUTION:*

$$\frac{\tan\frac{\pi}{9} + \tan\frac{8\pi}{9}}{1 - \tan\frac{\pi}{9}\tan\frac{8\pi}{9}} = \tan\left(\frac{\pi}{9} + \frac{8\pi}{9}\right)$$
$$= \tan\pi$$
$$= 0$$

52. $\cos 24° \cos 36° - \sin 24° \sin 36°$

*SOLUTION:*

$$\cos 24° \cos 36° - \sin 24° \sin 36° = \cos(24° + 36°) = \cos 60° = \frac{1}{2}$$

53. $\sin 95° \cos 50° - \cos 95° \sin 50°$

*SOLUTION:*

$$\sin 95° \cos 50° - \cos 95° \sin 50° = \sin(95° - 50°) = \sin 45° = \frac{\sqrt{2}}{2}$$

54. $\cos \frac{2\pi}{9} \cos \frac{\pi}{18} + \sin \frac{2\pi}{9} \sin \frac{\pi}{18}$

*SOLUTION:*

$$\cos\frac{2\pi}{9}\cos\frac{\pi}{18} + \sin\frac{2\pi}{9}\sin\frac{\pi}{18} = \cos\left(\frac{2\pi}{9} - \frac{\pi}{18}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

**Verify each identity.**

55. $\cos(\theta + 30°) - \sin(\theta + 60°) = -\sin \theta$

*SOLUTION:*

$$\begin{aligned}&\cos(\theta + 30°) - \sin(\theta + 60°)\\ &= \cos\theta\cos 30° - \sin\theta\sin 30° - (\sin\theta\cos 60° + \cos\theta\sin 60°)\\ &= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\\ &= -\sin\theta\end{aligned}$$

56. $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta)$

*SOLUTION:*

$$\begin{aligned}&\cos\left(\theta + \frac{\pi}{4}\right)\\ &= \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} && \text{Cosine Sum Identity}\\ &= \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta && \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \text{ and } \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta) && \text{Factor } \frac{\sqrt{2}}{2} \text{ from each term.}\end{aligned}$$

57. $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

*SOLUTION:*

$$\begin{aligned}&\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right)\\ &= \cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3} + \cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\\ &= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\\ &= \cos\theta\end{aligned}$$

58. $\tan\left(\theta + \frac{3\pi}{4}\right) = \frac{\tan\theta - 1}{\tan\theta + 1}$

*SOLUTION:*

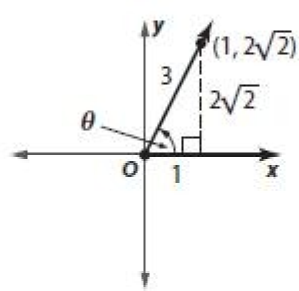
$$\begin{aligned}&\tan\left(\theta + \frac{3\pi}{4}\right)\\ &= \frac{\tan\theta + \tan\frac{3\pi}{4}}{1 - \tan\theta\tan\frac{3\pi}{4}} && \text{Tangent Sum Identity}\\ &= \frac{\tan\theta - 1}{1 - \tan\theta\cdot(-1)} && \tan\frac{3\pi}{4} = -1\\ &= \frac{\tan\theta - 1}{\tan\theta + 1} && \text{Simplify.}\end{aligned}$$

**Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.**

59. $\cos\theta = \frac{1}{3}$, $(0°, 90°)$

*SOLUTION:*

Since $\cos\theta = \frac{1}{3}$ on the interval (0°, 90°), one point on the terminal side of $\theta$ has $x$-coordinate 1 and a distance of 3 units from the origin as shown. The $y$-coordinate of this point is therefore $\sqrt{3^2 - 1^2}$ or $2\sqrt{2}$.




Using this point, we find that $\sin\theta = \frac{y}{r}$ or $\frac{2\sqrt{2}}{3}$ and

$\tan\theta = \frac{y}{x}$ or $2\sqrt{2}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2$\theta$, cos 2$\theta$, and tan 2$\theta$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right)\\ &= \frac{4\sqrt{2}}{9}\end{aligned}$$

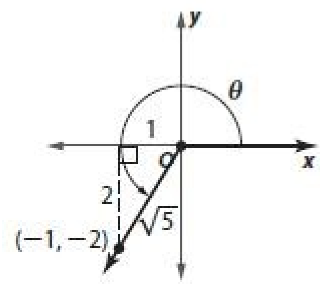
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{1}{3}\right)^2 - 1\\ &= 2\left(\frac{1}{9}\right) - 1\\ &= \frac{2}{9} - \frac{9}{9}\\ &= -\frac{7}{9}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2(2\sqrt{2})}{1-(2\sqrt{2})^2}\\ &= \frac{4\sqrt{2}}{1-8}\\ &= -\frac{4\sqrt{2}}{7}\end{aligned}$$

60. tan $\theta$ = 2, (180°, 270°)

*SOLUTION:*

If tan $\theta$ = 2, then tan $\theta$ = $\frac{2}{1}$. Since tan $\theta$ = $\frac{2}{1}$ on the interval (180°, 270°), one point on the terminal side of $\theta$ has $x$-coordinate −1 and $y$-coordinate −2 as shown. The distance from the point to the origin is $\sqrt{2^2+1^2}$ or $\sqrt{5}$.




Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{2\sqrt{5}}{5}$ and $\cos\theta = \frac{x}{r}$ or $-\frac{\sqrt{5}}{5}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2$\theta$, cos 2$\theta$, and tan 2$\theta$.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{2\sqrt{5}}{5}\right)\left(-\frac{\sqrt{5}}{5}\right)\\ &= \frac{4}{5}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{\sqrt{5}}{5}\right)^2 - 1\\ &= 2\left(\frac{5}{25}\right) - 1\\ &= \frac{10}{25} - \frac{25}{25}\\ &= -\frac{3}{5}\end{aligned}$$

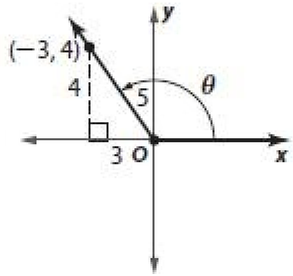
$$\begin{aligned}\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\\ &= \frac{2(2)}{1-(2)^2}\\ &= \frac{4}{1-4}\\ &= -\frac{4}{3}\end{aligned}$$

61. $\sin\theta = \frac{4}{5}, \left(\frac{\pi}{2}, \pi\right)$

*SOLUTION:*

Since $\sin\theta = \frac{4}{5}$ on the interval $\left(\frac{\pi}{2}, \pi\right)$, one point on the terminal side of $\theta$ has $y$-coordinate 4 and a distance of 5 units from the origin as shown. The $x$-

coordinate of this point is therefore $-\sqrt{5^2-4^2}$ or −3.




Using this point, we find that $\cos\theta = \frac{x}{r}$ or $-\frac{3}{5}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{4}{3}$. Now use the double-angle identities for sine and cosine to find sin 2*θ* and cos 2*θ*.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{3}{5}\right)^2 - 1\\ &= 2\left(\frac{9}{25}\right) - 1\\ &= \frac{18}{25} - \frac{25}{25}\\ &= -\frac{7}{25}\end{aligned}$$

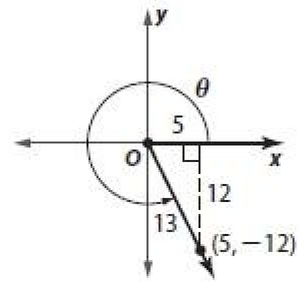
Use the definition of tangent to find tan 2*θ*.

$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta}\\ &= \frac{-\frac{24}{25}}{-\frac{7}{25}}\\ &= \frac{24}{7}\end{aligned}$$

62. $\sec\theta = \frac{13}{5}, \left(\frac{3\pi}{2}, 2\pi\right)$

*SOLUTION:*

If $\sec\theta = \frac{13}{5}$, then $\cos\theta = \frac{5}{13}$. Since $\cos\theta = \frac{5}{13}$ on the interval $\left(\frac{3\pi}{2}, 2\pi\right)$, one point on the terminal side of *θ* has *x*-coordinate 5 and a distance of 13 units from the origin as shown. The *y*-coordinate of this point is therefore $-\sqrt{13^2-5^2}$ or −12.




Using this point, we find that $\sin\theta = \frac{y}{r}$ or $-\frac{12}{13}$ and $\tan\theta = \frac{y}{x}$ or $-\frac{12}{5}$. Now use the double-angle identities for sine and cosine to find sin 2*θ* and cos 2*θ*.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{12}{13}\right)\left(\frac{5}{13}\right)\\ &= -\frac{120}{169}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(\frac{5}{13}\right)^2 - 1\\ &= 2\left(\frac{25}{169}\right) - 1\\ &= \frac{50}{169} - \frac{169}{169}\\ &= -\frac{119}{169}\end{aligned}$$

Use the definition of tangent to find tan 2*θ*.

$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta}\\ &= \frac{-\frac{120}{169}}{-\frac{119}{169}}\\ &= \frac{120}{119}\end{aligned}$$

**Find the exact value of each expression.**

63. $\sin 75°$

*SOLUTION:*

Notice that 75° is half of 150°. Therefore, apply the half-angle identity for sine, noting that since 75° lies in Quadrant I, its sine is positive.

$$\begin{aligned}\sin 75° &= \sin \frac{150°}{2}\\ &= \sqrt{\frac{1-\cos 150°}{2}}\\ &= \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}\\ &= \sqrt{\frac{2+\sqrt{3}}{4}}\\ &= \frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

64. $\cos \frac{11\pi}{12}$

*SOLUTION:*

Notice that $\frac{11\pi}{12}$ is half of $\frac{11\pi}{6}$. Therefore, apply the half-angle identity for cosine, noting that since $\frac{11\pi}{12}$ lies in Quadrant II, its cosine is negative.

$$\begin{aligned}\cos\frac{11\pi}{12} &= \cos\frac{\frac{11\pi}{6}}{2}\\ &= -\sqrt{\frac{1+\cos\frac{11\pi}{6}}{2}}\\ &= -\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= -\sqrt{\frac{2+\sqrt{3}}{4}}\\ &= -\frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

65. $\tan 67.5°$

*SOLUTION:*

Notice that 67.5° is half of 135°. Therefore, apply the half-angle identity for tangent, noting that since 67.5° lies in Quadrant I, its tangent is positive.

$$\begin{aligned}\tan 67.5° &= \tan\frac{135°}{2}\\ &= \frac{1-\cos 135°}{\sin 135°}\\ &= \frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}}\\ &= \frac{2+\sqrt{2}}{\sqrt{2}}\\ &= \frac{2\sqrt{2}+2}{2}\\ &= \sqrt{2}+1\end{aligned}$$

66. cos $\frac{3\pi}{8}$

*SOLUTION:*

Notice that $\frac{3\pi}{8}$ is half of $\frac{3\pi}{4}$. Therefore, apply the half-angle identity for cosine, noting that since $\frac{3\pi}{8}$ lies in Quadrant I, its cosine is positive.

$$\cos\frac{3\pi}{8} = \cos\frac{\frac{3\pi}{4}}{2}$$
$$= \sqrt{\frac{1+\cos\frac{3\pi}{4}}{2}}$$
$$= \sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}}$$
$$= \sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

67. sin $\frac{15\pi}{8}$

*SOLUTION:*

Notice that $\frac{15\pi}{8}$ is half of $\frac{15\pi}{4}$. Therefore, apply the half-angle identity for sine, noting that since $\frac{15\pi}{8}$ lies in Quadrant IV, its sine is negative.

$$\sin\frac{15\pi}{8} = \sin\frac{\frac{15\pi}{4}}{2}$$
$$= -\sqrt{\frac{1-\cos\frac{15\pi}{4}}{2}}$$
$$= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$
$$= -\sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= -\frac{\sqrt{2-\sqrt{2}}}{2}$$

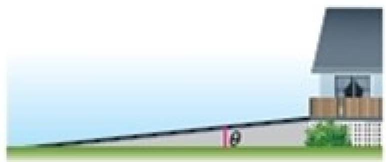
68. tan $\frac{13\pi}{12}$

*SOLUTION:*

Notice that $\frac{13\pi}{12}$ is half of $\frac{13\pi}{6}$. Therefore, apply the half-angle identity for tangent, noting that since $\frac{13\pi}{12}$ lies in Quadrant III, its tangent is positive.

$$\tan\frac{13\pi}{12} = \tan\frac{\frac{13\pi}{6}}{2}$$
$$= \frac{1-\cos\frac{13\pi}{6}}{\sin\frac{13\pi}{6}}$$
$$= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$
$$= 2-\sqrt{3}$$

69. **CONSTRUCTION** Find the tangent of the angle that the ramp makes with the building if sin $\theta$ = $\frac{\sqrt{145}}{145}$ and cos $\theta$ = $\frac{12\sqrt{145}}{145}$.




*SOLUTION:*

Use the Quotient Identity tan $\theta = \frac{\sin\theta}{\cos\theta}$.

| | |
|---|---|
| $\tan\theta = \frac{\sin\theta}{\cos\theta}$ | Quotient Identity |
| $= \frac{\frac{\sqrt{145}}{145}}{\frac{12\sqrt{145}}{145}}$ | $\sin\theta = \frac{\sqrt{145}}{145}$; $\cos\theta = \frac{12\sqrt{145}}{145}$ |
| $= \frac{\sqrt{145}}{145}\cdot\frac{145}{12\sqrt{145}}$ | Multiply by the reciprocal. |
| $= \frac{1}{12}$ | Simplify. |

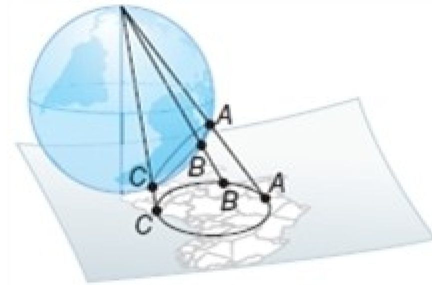
Therefore, the tangent of the angle the ramp makes with the building is $\frac{1}{12}$.

70. **LIGHT** The intensity of light that emerges from a system of two polarizing lenses can be calculated by $I = I_0 - \frac{I_0}{\csc^2\theta}$, where $I_0$ is the intensity of light entering the system and $\theta$ is the angle of the axis of the second lens with the first lens. Write the equation for the light intensity using only tan $\theta$.

*SOLUTION:*

| | |
|---|---|
| $I = I_0 - \frac{I_0}{\csc^2\theta}$ | Original equation |
| $= I_0 - \frac{I_0}{\cot^2\theta + 1}$ | Pythagorean Identity |
| $= I_0 - \frac{I_0}{\frac{1}{\tan^2\theta} + 1}$ | Reciprocal Identity |
| $= I_0 - \frac{I_0}{\frac{1}{\tan^2\theta} + \frac{\tan^2\theta}{\tan^2\theta}}$ | Rewrite 1 as $\frac{\tan^2\theta}{\tan^2\theta}$. |
| $= I_0 - \frac{I_0}{\frac{1+\tan^2\theta}{\tan^2\theta}}$ | Add. |
| $= I_0 - I_0\left(\frac{\tan^2\theta}{1+\tan^2\theta}\right)$ | Multiply by the reciprocal. |

71. **MAP PROJECTIONS** Stereographic projection is used to project the contours of a three-dimensional sphere onto a two-dimensional map. Points on the sphere are related to points on the map using $r = \frac{\sin\alpha}{1-\cos\alpha}$. Verify that $r = \frac{1+\cos\alpha}{\sin\alpha}$.




*SOLUTION:*

Verify that $\frac{\sin\alpha}{1-\cos\alpha} = \frac{1+\cos\alpha}{\sin\alpha}$.

| | |
|---|---|
| $\frac{\sin\alpha}{1-\cos\alpha}$ | |
| $= \frac{\sin\alpha}{1-\cos\alpha} \cdot \frac{1+\cos\alpha}{1+\cos\alpha}$ | Multiply numerator and denominator by $1+\cos\alpha$. |
| $= \frac{\sin\alpha(1+\cos\alpha)}{1-\cos^2\alpha}$ | Multiply. |
| $= \frac{\sin\alpha(1+\cos\alpha)}{\sin^2\alpha}$ | Pythagorean Identity |
| $= \frac{1+\cos\alpha}{\sin\alpha}$ | Divide the numerator and denominator by $\sin\alpha$. |

By substitution, $r = \frac{1+\cos\alpha}{\sin\alpha}$.

72. **PROJECTILE MOTION** A ball thrown with an initial speed of $v_0$ at an angle $\theta$ that travels a horizontal distance $d$ will remain in the air $t$ seconds, where $t = \frac{d}{v_0 \cos\theta}$. Suppose a ball is thrown with an initial speed of 50 feet per second, travels 100 feet, and is in the air for 4 seconds. Find the angle at which the ball was thrown.

*SOLUTION:*

Let $t = 4$, $d = 100$, $v_0 = 50$, and solve for $\theta$.

$$t = \frac{d}{v_0 \cos\theta}$$
$$4 = \frac{100}{50\cos\theta}$$
$$4(50\cos\theta) = 100$$
$$50\cos\theta = 25$$
$$\cos\theta = \frac{25}{50}$$
$$\cos\theta = \frac{1}{2}$$

On the unit circle, $\cos\theta = \frac{1}{2}$ when $\theta = 60°$ and $\theta = 300°$. Because the inverse cosine function is restricted to acute angles of $\theta$ on the interval [0, 180°], $\theta = 60°$. Therefore, the ball was thrown at an angle of 60°.

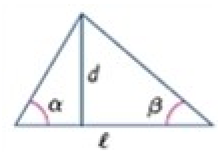
73. **BROADCASTING** Interference occurs when two waves pass through the same space at the same time. It is destructive if the amplitude of the sum of the waves is less than the amplitudes of the individual waves. Determine whether the interference is destructive when signals modeled by $y = 20 \sin(3t + 45°)$ and $y = 20 \sin(3t + 225°)$ are combined.

*SOLUTION:*

$$20\sin(3t+45°) + 20\sin(3t+225°)$$
$$= 20(\sin 3t \cos 45° + \cos 3t \sin 45°) + 20(\sin 3t \cos 225° + \cos 3t \sin 225°)$$
$$= 20\left[\sin 3t\left(\frac{\sqrt{2}}{2}\right) + \cos 3t\left(\frac{\sqrt{2}}{2}\right)\right] + 20\left[\sin 3t\left(-\frac{\sqrt{2}}{2}\right) + \cos 3t\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$= 20\sin 3t\left(\frac{\sqrt{2}}{2}\right) + 20\cos 3t\left(\frac{\sqrt{2}}{2}\right) - 20\sin 3t\left(\frac{\sqrt{2}}{2}\right) - 20\cos 3t\left(\frac{\sqrt{2}}{2}\right)$$
$$= 0$$

The sum of the two functions is zero, which means that the amplitude is 0. Because the amplitude of each of the original functions was 20 and 0 < 20, the combination of the two waves can be characterized as producing destructive interference.

74. **TRIANGULATION** *Triangulation* is the process of measuring a distance $d$ using the angles $\alpha$ and $\beta$ and the distance $\ell$ using $\ell = \frac{d}{\tan\alpha} \frac{d}{\tan\beta}$.




**a.** Solve the formula for $d$.

**b.** Verify that $d = \frac{\ell \sin\alpha \sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$.

**c.** Verify that $d = \frac{\ell \sin\alpha \sin\beta}{\sin(\alpha+\beta)}$.

**d.** Show that if $\alpha = \beta$, then $d = 0.5\ell \tan\alpha$

*SOLUTION:*

**a.**

$$\ell = \frac{d}{\tan\alpha} + \frac{d}{\tan\beta}$$
$$\ell = d\left[\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}\right]$$
$$\frac{\ell}{\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}} = d$$

**b.**

$$\frac{\ell}{\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}}$$

$$=\frac{\ell}{\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}}$$

$$=\frac{\ell}{\frac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}+\frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\ell}{\frac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}$$

$$=\ell\cdot\frac{\sin\alpha\sin\beta}{\cos\alpha\sin\beta+\sin\alpha\cos\beta}$$

$$=\frac{\ell\sin\alpha\sin\beta}{\cos\alpha\sin\beta+\sin\alpha\cos\beta}$$

**c.**

$$\frac{\ell\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

$$=\frac{\ell\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$ Sine Sum Identity

**d.**

$$\frac{\ell\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

$$=\frac{\ell\sin\alpha\sin\alpha}{\sin(\alpha+\alpha)}$$ Let $\alpha=\beta$.

$$=\frac{\ell\sin^2\alpha}{\sin2\alpha}$$ Simplify.

$$=\frac{\ell\sin^2\alpha}{2\sin\alpha\cos\alpha}$$ Sine Double-Angle Identity

$$=\frac{\ell\sin\alpha}{2\cos\alpha}$$ Divide numerator and denominator by $\sin\alpha$.

$$=0.5\ell\tan\alpha$$ Quotient Identity