7-2 Ellipses and Circles

Write each equation in standard form. Identify the related conic.

25. $4x^2 + 8y^2 - 8x + 48y + 44 = 0$

SOLUTION:

$4x^2 + 8y^2 - 8x + 48y + 44 = 0$

$4x^2 - 8x + 8y^2 + 48y + 4 = 0$

$4(x^2 - 2x + 1) + 8(y^2 + 6y + 9) + 44 = 0 + 4 + 72$

$4(x - 1)^2 + 8(y + 3)^2 = 32$

$$\frac{(x-1)^2}{8} + \frac{(y+3)^2}{4} = 1$$

The related conic is an ellipse because $a \neq b$ and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$

27. $y^2 - 12x + 18y + 153 = 0$

SOLUTION:

$y^2 - 12x + 18y + 153 = 0$

$y^2 + 18y + 153 = 12x$

$y^2 + 18y + 81 + 153 = 12x + 81$

$(y + 9)^2 = 12x - 72$

$(y + 9)^2 = 12(x - 6)$

Because only one term is squared, the graph is a parabola.

29. $3x^2 + y^2 - 42x + 4y + 142 = 0$

SOLUTION:

$3x^2 + y^2 - 42x + 4y + 142 = 0$

$3x^2 - 42x + y^2 + 4y + 142 = 0$

$3(x^2 - 14x + 49) + y^2 + 4y + 4 + 142 = 0 + 147 + 4$

$3(x - 7)^2 + (y + 2)^2 = 9$

$$\frac{(x-7)^2}{3} + \frac{(y+2)^2}{9} = 1$$

The related conic is an ellipse because $a \neq b$ and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$

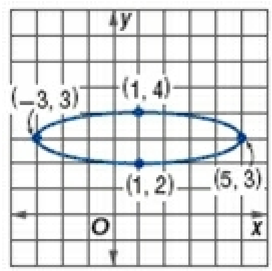
31. $2x^2 + 7y^2 + 24x + 84y + 310 = 0$

SOLUTION:

$$2x^2 + 7y^2 + 24x + 84y + 310 = 0$$
$$2x^2 + 24x + 7y^2 + 84y + 310 = 0$$
$$2\left(x^2 + 12x + 36\right) + 7\left(y^2 + 12y + 36\right) + 310 = 0 + 72 + 252$$
$$2(x+6)^2 + 7(y+6)^2 = 14$$
$$\frac{(y+6)}{2} + \frac{(x+6)}{7} = 1$$

The related conic is an ellipse because $a \neq b$, and the graph is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$

Write an equation for each ellipse.

39. 

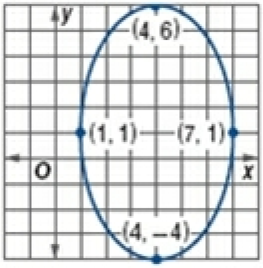
SOLUTION:

From the graph, you can see that the vertices are located at (–3, 3) and (5, 3) and the co–vertices are located at (1, 4) and (1, 2). The major axis is horizontal, so the standard form of the equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

The center is the midpoint of the segment between the vertices, or (1, 3). So, $h = 1$ and $k = 3$.
The distance between the vertices is equal to $2a$ units. So, $2a = 8$, $a = 4$, and $a^2 = 16$. The distance between the co–vertices is equal to $2b$ units. So, $2b = 2$, $b = 1$, and $b^2 = 1$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-1)^2}{16} + \frac{(y-3)^2}{1} = 1.$

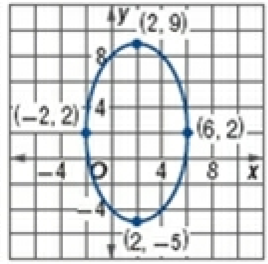
40. 

SOLUTION:

From the graph, you can see that the vertices are located at (4, 6) and (4, –4) and the co–vertices are located at (1, 1) and (7, 1). The major axis is vertical, so the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is the midpoint of the segment between the vertices, or (4, 1). So, $h = 4$ and $k = 1$.
The distance between the vertices is equal to $2a$ units. So, $2a = 10$, $a = 5$, and $a^2 = 25$. The distance between the co–vertices is equal to $2b$ units. So, $2b = 6$, $b = 3$, and $b^2 = 9$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-4)^2}{9}+\frac{(y-1)^2}{25}=1$.

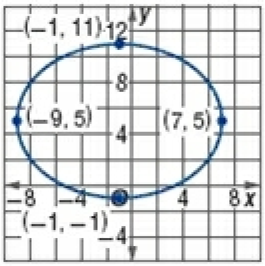
41. 

SOLUTION:

From the graph, you can see that the vertices are located at (2, 9) and (2, –5) and the co–vertices are located at (–2, 2) and (6, 2). The major axis is vertical, so the standard form of the equation is $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

The center is the midpoint of the segment between the vertices, or (2, 2). So, $h = 2$ and $k = 2$.
The distance between the vertices is equal to $2a$ units. So, $2a = 14$, $a = 7$, and $a^2 = 49$. The distance between the co–vertices is equal to $2b$ units. So, $2b = 8$, $b = 4$, and $b^2 = 16$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-2)^2}{16}+\frac{(y-2)^2}{49}=1$.

42. 

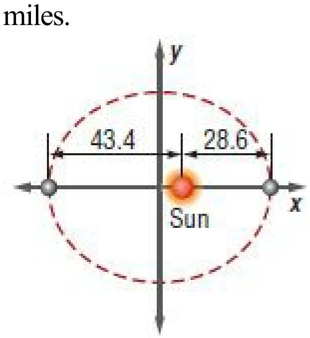
SOLUTION:

From the graph, you can see that the vertices are located at (–9, 5) and (7, 5) and the co–vertices are located at (–1, 11) and (–1, 1). The major axis is horizontal, so the standard form of the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

The center is the midpoint of the segment between the vertices, or (–1, 5). So, $h = -1$ and $k = 5$. The distance between the vertices is equal to $2a$ units. So, $2a = 16$, $a = 8$, and $a^2 = 64$. The distance between the co–vertices is equal to $2b$ units. So, $2b = 12$, $b = 6$, and $b^2 = 36$.

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x+1)^2}{64} + \frac{(y-5)^2}{36} = 1$.

43. **PLANETARY MOTION** Each of the planets in the solar system move around the Sun in an elliptical orbit, where the Sun is one focus of the ellipse. Mercury is 43.4 million miles from the Sun at its farthest point and 28.6 million miles at its closest, as shown below. The diameter of the Sun is 870,000 miles.



a. Find the length of the minor axis.
b. Find the eccentricity of the elliptical orbit.

SOLUTION:

a. The length of the major axis is 43.4 + 28.6 + 0.87 = 72.87.

The value of a is 72.87 ÷ 2 = 36.435. The distance from the focus (the sun) to the vertex is $28.6 + \frac{0.87}{2} = 29.035$.

Therefore, the value of c, the focus to the center, is 36.435 − 29.035 = 7.4.

In an ellipse, $c^2 = a^2 - b^2$. Use the values of a and c to find b.

$$c^2 = a^2 - b^2$$
$$7.4^2 = 36.435^2 - b^2$$
$$36.435^2 - 7.4^2 = b^2$$
$$\sqrt{36.435^2 - 7.4^2} = b$$
$$35.676 = b$$

The value of $2b$, the length of the minor axis, is about $35.676 \cdot 2 \approx 71.35$ million miles.

b. Use the values of a and c to find the eccentricity of the orbit.

$$e = \frac{c}{a} = \frac{7.4}{36.435} \approx 0.203$$

So, the orbit has an eccentricity of about 0.203.

7-2 Ellipses and Circles

Find the center, foci, and vertices of each ellipse.

47. $65x^2 + 16y^2 + 130x - 975 = 0$

SOLUTION:

First, write the equation in standard form.

$$65x^2 + 16y^2 + 130x - 975 = 0$$
$$65x^2 + 130x + 16y^2 - 975 = 0$$
$$65\left(x^2 + 2x + 1\right) + 16y^2 = 0 + 975 + 65$$
$$65(x+1)^2 + 16y^2 = 1040$$
$$\frac{(x+1)^2}{16} + \frac{y^2}{65} = 1$$

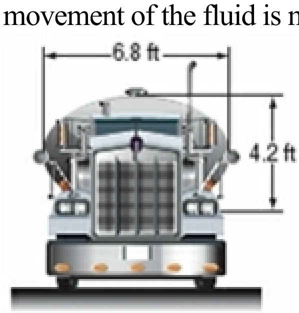
The equation is now in standard form, where $h = -1$ and $k = 0$. So, the center is located at (–1, 0). The ellipse has a vertical orientation, so $a^2 = 65$, $a = \sqrt{65}$, and $b^2 = 16$.

Use the values of a and b to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 65 - 16$$
$$c^2 = 49$$
$$c = 7$$

The foci are c units from the center, so they are located at (–1, ±7). The vertices are a units from the center, so they are located at $(-1, \pm\sqrt{65})$.

48. **TRUCKS** Elliptical tanker trucks like the one shown are often used to transport liquids because they are more stable than circular tanks and the movement of the fluid is minimized.



a. Draw and label the elliptical cross–section of the tank on a coordinate plane.

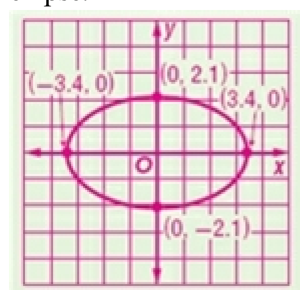
b. Write an equation to represent the elliptical shape of the tank.

c. Find the eccentricity of the ellipse.

SOLUTION:

a. The major axis of the ellipse is 6.8 ft, so $a = 3.4$. The minor axis is 4.2 ft, so $b = 2.1$. The ellipse is centered about the origin, so $h = k = 0$. The major axis is along the x–axis, so the ellipse is horizontal. The vertices are located at (0 ± 3.4, 0) and the co–vertices are located at (0, 0 ± 2.1).

Plot the vertices and co–vertices to sketch the ellipse.



b. The orientation of the ellipse is horizontal, so the standard form of the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Substitute the values of a and b into the equation.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$\frac{(x-0)^2}{3.4^2} + \frac{(y-0)^2}{2.1^2} = 1$$
$$\frac{x^2}{11.56} + \frac{y^2}{4.41} = 1$$

c. Use the values of a and b to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 11.56 - 4.41$$
$$c^2 = 7.15$$
$$c \approx 2.67$$

Substitute a and c into the eccentricity equation.

$$
\begin{aligned}
e &= \frac{c}{a} \\
&= \frac{2.67}{3.4} \\
&\approx 0.79
\end{aligned}
$$

So, the eccentricity of the ellipse is about 0.79.